Introduction to Matching theory

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Table of contents

1 Introduction

2 Theory

3 Matching

4 Stable Matching
What is market design

- Traditional economics takes the economy as it is.
- Nowadays economists are using economics insight to design markets and institutions. Examples are 1) Labor market, matching doctors to hospitals 2) Student placement to schools. 3) Allocating courses to students.
- Example 1 and 2 are a two sided matching problem in which both sides have preference over the other side.
- Example 3 is one sided in which one side has preferences over the other side.
- Some markets that were operating freely failed and were successfully re-designed. Is market design a government intervention?
The model was proposed by Gale and Shapley (1962).

Finite sets $S$ of students and $C$ of colleges (we use student-college terminology just for convenience).

Each student can be matched to at most one college, and each college can admit at most one student (so the model is called one-to-one matching).

Students have strict preferences over colleges and being unmatched (denoted by $\emptyset$) and colleges have strict preferences over students and being unmatched.

- $c \succ_s c'$ means student $s$ strictly prefers college $c$ to college $c'$.
- $s \succ_c s'$ means college $c$ strictly prefers student $s$ to student $s'$.

If $i \succ_j \emptyset$ then we say $i$ is acceptable to $j$. 
Example

There are three students \( \{s_1, s_2, s_3\} \) and three colleges \( \{c_1, c_2, c_3\} \) with the following preferences:

1. \( c_2 \succ s_1 \), \( c_1 \succ s_1 \), \( \emptyset \succ s_1 \), \( c_3 \)
2. \( c_1 \succ s_2 \), \( c_2 \succ s_2 \), \( c_3 \succ s_2 \), \( \emptyset \)
3. \( c_1 \succ s_3 \), \( c_3 \succ s_3 \), \( \emptyset \succ s_3 \), \( c_2 \)
4. \( s_2 \succ c_1 \), \( s_1 \succ c_1 \), \( \emptyset \succ c_1 \), \( s_3 \)
5. \( s_1 \succ c_2 \), \( s_3 \succ c_2 \), \( \emptyset \succ c_2 \), \( s_2 \)
6. \( s_2 \succ c_3 \), \( s_1 \succ c_3 \), \( \emptyset \succ c_3 \), \( s_3 \)
The outcome of the matching market is a matching, which species which student attends which college.

Formally, matching is a function from $S \cup C$ to $S \cup C \cup \{\emptyset\}$ such that:

1. $\mu(s) \in C \cup \{\emptyset\}$
2. $\mu(c) \in S \cup \{\emptyset\}$
3. $\mu(s) = c \iff \mu(c) = s$
For example $\mu(s_1) = c_1$, $\mu(s_2) = c_2$, $\mu(s_3) = \emptyset$ and $\mu(c_3) = \emptyset$ is a matching in which student 3 and college 3 are unmatched.
Roughly speaking, a matching is **stable** if there is no individual players or pairs of players who can profitably deviate from (block) it.

Matching is **blocked** by an individual $i$ if $\mu(i)$ is unacceptable to $i$, that is $\emptyset \succ_i \mu(i)$.

Matching is blocked by a pair $s$ and $c$ if each of them prefer each other to their partners under $\mu$ that is: $c \succ_s \mu(c)$ and $s \succ_c \mu(s)$.
Definition of stability

- Roughly speaking, a matching is **stable** if there is no individual players or pairs of players who can profitably deviate from (block) it.
- Matching is **blocked** by an individual $i$ if $\mu(i)$ is unacceptable to $i$, that is $\emptyset \succ_i \mu(i)$.
- Matching is blocked by a pair $s$ and $c$ if each of them prefer each other to their partners under $\mu$ that is: $c \succ_s \mu(c)$ and $s \succ_c \mu(s)$.

A matching is stable if it is not blocked by any individual or pair
The matching in the previous example is not stable. $s_1$ and $c_2$ create a block since.

$s_1 \succ_{c_2} \mu(c_2) = s_2$ and $c_2 \succ_{s_1} \mu(s_1) = s_1$. 
Example

The matching in the previous example is not stable. \( s_1 \) and \( c_2 \) create a block since:

\[ s_1 \succ c_2 \mu(c_2) = s_2 \text{ and } c_2 \succ s_1 \mu(s_1) = s_1. \]

The following is a stable match in that example.

\[ \mu(s_1) = c_2, \mu(s_2) = c_1, \mu(s_3) = \emptyset \text{ and } \mu(c_3) = \emptyset \]
Gale and Shapley 1962 proposed an algorithm, called DA, that produce a stable match.

Gale passed away but Shapley got the Nobel prize this year.

Their mechanism or a variation of their mechanism is widely used in practice.

Why should we care about stability?
Rural Hospital theorem

Set of unmatched colleges and students are the same in all stable matchings.
References


- Roth, Alvin E. and Marilda Sotomayor [1990], Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis, Cambridge, Cambridge University Press.
