Introduction to Matching theory

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What is market design

- Traditional economics takes the economy as it is.
- Nowadays economists are using economics insight to **design markets and institutions**. Examples are 1) Labor market, matching doctors to hospitals 2) Student placement to schools. 3) Allocating courses to students.
- Example 1 and 2 are a two sided matching problem in which both sides have preference over the other side.
- Example 3 is one sided in which one side has preferences over the other side.
- Some markets that were operating freely failed and were successfully re-designed. Is market design a government intervention?

Student-College model

- The model was Proposed by Gale and Shapley (1962).
- Finite sets S of **students** and C of **colleges** (we use student-college terminology just for convenience).
- Each student can be **matched** to at most one college, and each college can admit at most one student (so the model is called one-to-one matching).
- Students have strict preferences over colleges and being **unmatched** (denoted by $\emptyset$) and colleges have **strict preferences** over students and being unmatched.
- $c \succ_s c'$ means student s strictly prefers college c to college c' .
- $s \succ_c s'$ means college c strictly prefers student s to student s' . If $i \succ_j \emptyset$ then we say i is acceptable to j.

Example

- There are three students $\{s_1, s_2, s_3\}$ and three colleges $\{c_1, c_2, c_3\}$ with the following preferences

1. $c_2 \succ_{s_1} c_1 \succ_{s_1} \emptyset \succ_{s_1} c_3$
2. $c_1 \succ_{s_2} c_2 \succ_{s_2} c_3 \succ_{s_2} \emptyset$
3. $c_1 \succ_{s_3} c_3 \succ_{s_3} \emptyset \succ_{s_3} c_2$
4. $s_2 \succ_{c_1} s_1 \succ_{c_1} \emptyset \succ_{c_1} s_3$
5. $s_1 \succ_{c_2} s_3 \succ_{c_2} \emptyset \succ_{c_2} s_2$
6. $s_2 \succ_{c_3} s_1 \succ_{c_3} \emptyset \succ_{c_3} s_3$

Matching

- The outcome of the **matching market** is a **matching**, which species which student attends which college.
- Formally, matching is a function from $S \cup C$ to $S \cup C \cup \{\emptyset\}$ such that:

1. $\mu(s) \in C \cup \{\emptyset\}$
2. $\mu(c) \in S \cup \{\emptyset\}$
3. $\mu(s) = c \iff \mu(c) = s$

Example

For example $\mu(s_1) = c_1$, $\mu(s_2) = c_2$, $\mu(s_3) = \emptyset$ and $\mu(c_3) = \emptyset$ is a matching in which student 3 and college 3 are unmatched.

Definition of stability

- Roughly speaking, a matching is **stable** if there is no individual players or pairs of players who can profitably deviate from (**block**) it.
- Matching is **blocked** by an individual i if $\mu(i)$ is unacceptable to i, that is $\emptyset \succ_i \mu(i)$.
- Matching is blocked by a pair s and c if each of them prefer each other to their partners under μ that is: $c \succ_s \mu(c)$ and $s \succ_c \mu(s)$.

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A matching is stable if it is not blocked by any individual or pair

Example

The matching in the previous example is not stable. s_1 and c_2 create a block since.
$s_1 \succ_{c_2} \mu(c_2) = s_2$ and $c_2 \succ_{s_1} \mu(s_1) = s_1$.

Example

The matching in the previous example is not stable. s_1 and c_2 create a block since.
$s_1 \succ_{c_2} \mu(c_2) = s_2$ and $c_2 \succ_{s_1} \mu(s_1) = s_1$.
The following is a stable match in that example.
$\mu(s_1) = c_2$, $\mu(s_2) = c_1$, $\mu(s_3) = \emptyset$ and $\mu(c_3) = \emptyset$

Remarkes

- Gale and Shapley 1962 proposed an algorithm, called **DA,** that produce a stable match.
- Gale passed away but Shapley got the Nobel prize this year.
- Their mechanism or a variation of their mechanism is widely used in practice.
- Why should we care about stability?

Rural Hospital theorem

Set of unmatched colleges and students are the same in all stable matchings.

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