Basics of Asset Pricing Theory

Continuous time consumption-portfolio choice

Yashar Heydari

University of Illinois

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Continuous time dynamic programing

- The problem is to maximize the integral of some objective function over a specific time horizon. The objective is a function of some control variable whose stream of optimal levels is chosen by the planner and a state variable that follows a diffusion process.

$$\max_{c_t} E_t[\int_t^T U(c_s, x_s)ds]$$

subject to

$$dx = a(x, c)dt + b(x, c)dz$$

- Can define the indirect utility (value) function as:

$$J(x_t, t) = \max_{x} E_t[\int_t^T U(c_s, x_s)ds]$$

$$= \max_{c} E_t[\int_t^{t+\Delta t} U(c_s, x_s)ds + \int_{t+\Delta t}^T U(c_s, x_s)ds]$$

Continuous time dynamic programing

- Distributing the max operator inside the brackets,

$$J(x_t, t) = \max_c E_t[\int_t^{t+\Delta t} U(c_s, x_s)ds + J(x_{t+\Delta t}, t + \Delta t)]$$

- Using a Taylor series expansion and employing diffusion processes properties, one can show that the problem reduces to:

$$0 = \max_c[U(c_t, x_t) + J_t + J_x a + \frac{1}{2} J_{xx} b^2]$$

$$= \max_c[U(c_t, x_t) + L[J]]$$

- For any dynamic programing problem, just need to identify functions and relationships correctly and apply the above result.

Continuous time consumption-portfolio problem - Merton (1969)

Assumptions

- There are n risky assets that pay no cash, each with a instantaneous return of:
$$\frac{dS_i}{S_i} = \mu_i(x,t)dt + \sigma_i(x,t)dz_i, \qquad i = 1, ..., n$$
and $\sigma_i dz_i \sigma_j dz_j = \sigma_{ij} dt$
- Risk free rate is given by $r(x,t)$ and consumption per unit time is c_t.
- There are k state variables summarized in vector x.
- Given variable μ_i's and σ_i's, the consumer faces changing investment opportunities.
- State variables can themselves follow diffusion processes:
$$dx_i = a_i(x,t)dt + b_i(x,t)d\zeta_i$$
with $b_i d\zeta_i c_j d\zeta_j = b_{ij} dt$ and $\sigma_i dz_i b_j d\zeta_j = \phi_{ij} dt$.

Continuous time consumption-portfolio problem

Problem

- The consumer's problem is given by:

$$\max_{C_s,\{\omega_{i,s}\}} E_t[\int_t^T U(C_s,s)ds + B(W_T,T)]$$

subject to

$$dW = [\sum_{i=1}^{n} \omega_i \frac{dS_i}{S_i} + (1-\sum_{i=1}^{n}\omega_i)r]W - Cdt$$

$$= \sum_{i=1}^{n} \omega_i(\mu_i - r)Wdt + (rW - C)dt + \sum_{i=1}^{n}\omega_i W \sigma_i dz_i$$

- In which both utility and bequest functions are strictly increasing and concave in the levle of consumption.

Solving the problem

- Define the indirect utility as:

$$J(W,x,t) = \max_{C_s,\{\omega_{i,s}\}} E_t\left[\int_t^T U(C_s,s)ds + B(W_T,T)\right]$$

- The Bellman equation will look like:

$$0 = \max_{C_t,\{\omega_{i,t}\}} \Big[U(C_t,t) + \frac{\partial J}{\partial t} + \Big[\sum_{i=1}^{n}\omega_i(\mu_i - r)W + (rW - C)\Big]\frac{\partial J}{\partial W}$$

$$+ \sum_{i=1}^{n} a_i \frac{\partial J}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}\omega_i\omega_j W^2 \frac{\partial^2 J}{\partial W^2}$$

$$+ \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k} b_{ij}\frac{\partial^2 J}{\partial x_i \partial x_j} + \sum_{j=1}^{k}\sum_{i=1}^{n} W\omega_i\phi_{ij}\frac{\partial^2 J}{\partial W \partial x_j}\Big]$$

Solving the problem

- FOC's:

$$\frac{\partial U(C^*, t)}{\partial C} = \frac{\partial J(W, x, t)}{\partial W}$$

$$0 = \frac{\partial J}{\partial W}(\mu_i - r)W + \frac{\partial^2 J}{\partial W^2}\sum_{j=1}^{k}\sigma_{ij}\omega_j^* W^2 + \frac{\partial^2 J}{\partial x_i \partial W}\sum_{j=1}^{k} W\phi_{ij}$$

- Defining $C^* = G(J_W, t) \Rightarrow \ C^* = G(J_W, t)$.
- Letting $\Omega = [\sigma_{ij}]$ and $\Omega^{-1} \equiv [\nu_{ij}]$,

$$\omega_i^* = -\frac{J_W}{J_{WW}W}\sum_{j=1}^{n}\nu_{ij}(\mu_j - r) - \sum_{m=1}^{k}\sum_{j=1}^{n}\frac{J_{W_{x_m}}}{J_{WW}W}\nu_{ij}\phi_{ij}$$

- Substituting back the optimal demands into Bellman equation, the PDE whose solution is the J function results.

Continuous time consumption-portfolio problem

Constant investment opportunities

- Constant investment opportunities is equivalent to μ_is and σ_is being constant.
- This leads to $\phi_{ij} = 0$ and the second term in the optimal portfolio choice condition vanishes.

$$\omega_i^* = -\frac{J_W}{J_{WW} W} \sum_{j=1}^{n} \nu_{ij}(\mu_i - r), \quad i = 1, ..., n$$

- Plugging back into the Bellman equation the PDE for the J function is found as:

$$0 = U(G, t) + J_t + J_W(rW - G) - \frac{J_W^2}{2J_{WW}} \sum_{i=1}^{n} \sum_{j=1}^{n} \nu_{ij}(\mu_i - r)(\mu_j - r)$$

- This doesn't have necessarily an easy/analytic solution.

Constant investment opportunities

- According to the results the proportion of wealth in risky asset i to risky asset j is constant:

$$\frac{\omega_i^*}{\omega_j^*} = \frac{\sum_{j=1}^{n} \nu_{ij}(\mu_{ij} - r)}{\sum_{j=1}^{n} \nu_{kj}(\mu_j - r)}$$

- The proportion of risky asset k to all risky assets is:

$$\delta_k = \frac{\omega_k^*}{\sum_{i=1}^{n} \omega_i^*} = \frac{\sum_{j=1}^{n} \nu_{kj}(\mu_j - r)}{\sum_{i=1}^{n} \sum_{j=1}^{n} \nu_{ij}(\mu_j - r)}$$

- Regardless of the form of the utility function, the individual holds a risk free asset and a portfolio which contains n risky assets in constant proportions.
- Investing in two "mutual funds" would satisfy investment needs of the individual.

Constant investment opportunities

- One can think of the portfolio decision in a constant investment opportunities setting to be investing in a risk-free asset and a market portfolio which possesses the following parameters:

$$\mu \equiv \sum_{i=1}^{n} \delta_i \mu_i$$

$$\sigma^2 \equiv \sum_{i=1}^{n} \sum_{j=1}^{n} \delta_i \delta_j \sigma_{ij}$$

- This is an equivalence condition which is derived due to the fact that choice of risky asset weights are such that relative weights are constant.

The Martingale approach

- Given that necessary conditions for the market being complete one can restate the problem based on a martingale approach.
- Recall from previous analysis regarding derivatives pricing that martingales exist if the market is frictionless - no-arbitrage condition holds.
- The ultimate and main result that assure market completeness in to prove that there exists a unique market price of risk.
- Through the martingale approach, quantities are discounted to present values using the stochastic discount factor.

The Martingale approach

- The budget constraint can be rewritten as:

$$W_t = E_t[\int_t^T \frac{M_T}{M_t} C_s ds + \frac{M_T}{M_t} W_t]$$

- Can use the Lagrange multiplier to restate the optimization problem.

$$\max_{C_s, W_T} E_t[\int_t^T U(C_s, s)ds + B(W_T, T)] + \lambda(M_t W_t - E_t[\int_t^T M_s C_s ds + M_T W_T])$$

- FOC's:

$$\frac{\partial U(C_s, s)}{\partial C_s} = \lambda M_s, \qquad \forall s \in [t, T]$$

$$\frac{\partial B(W_T, T)}{\partial W_T} = \lambda M_T$$

ICAPM

- Can use the FOC's to find $\mu_i - r$ for given asset.
- The use the results:

$$\frac{\omega_i^*}{\omega_j^*} = \frac{\sum_{j=1}^n \nu_{ij}(\mu_{ij} - r)}{\sum_{j=1}^n \nu_{kj}(\mu_k j - r)}$$

is constant.
And

$$\delta_k = \frac{\omega_k^*}{\sum_{i=1}^n \omega_i^*} = \frac{\sum_{j=1}^n \nu_{kj}(\mu_j - r)}{\sum_{i=1}^n \sum_{j=1}^n \nu_{ij}(\mu_j - r)}$$

- Also we know that there exist a portfolio delivering:

$$\mu \equiv \sum_{i=1}^n \delta_i \mu_i$$

$$\sigma^2 \equiv \sum_{i=1}^n \sum_{j=1}^n \delta_i \delta_j \sigma_{ij}$$

Continuous time consumption-portfolio problem - Merton (1969)

ICAPM

- Assume the portfolio generating expected return of μ is the market portfolio - $\mu = \mu_m$
- Use FOC's to derive $\mu_i - r$ for a given risky asset.
- Find a weighted average of $\mu_i - r$ according to δ_i which was computed earlier. This will be the market portfolio.
- With a few simple algebraic steps you can show that:

$$(\mu_i - r) = \beta_i(\mu_m - i)$$

where,

$$\beta_i = \frac{\sigma_{im}}{\sigma_m^2}$$

- This is what we've seen in discrete-time modeling.

Alternative preferences - Time-Inseparable Utility

Internal habit

- The consumer maximizes:

$$E_0[\int_0^\infty e^{-\rho t u(\hat{C}_t) dt}]$$

with $\hat{C}_t = C_t - bx_t$, and

$$x_t = e^{-at} x_0 + \int_0^t e^{-a(t-s)} C_s ds$$

- Consumer gets utility from consumption in excess of the habit level.
- Can have consumers who gain the same utility even though they consumption levels are different

Alternative preferences - Time-Inseparable Utility

External habit

- The consumer maximizes:

$$E_0[\sum_{t=0}^{\infty} \delta^t \frac{(C_t - X_t)^\gamma - 1}{\gamma}]$$

- Consumer compares her wellbeing to the average that she observes in the economy.
- Compare rich and poor countries