Basics of Asset Pricing Theory {Derivatives pricing - Martingales and pricing kernels

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Motivation

- In pricing contingent claims, it is common not to have a simple and traceable equilibrium PDE. $\Rightarrow$ Not easy to find the functional form of the price.
- Numerical methods? $\Rightarrow$ Not accurate, less interesting from theorists' point of view.
- What else?
- It can be shown that under the no-arbitrage condition, two alternative approaches could help:
  1. **Can use the martingale approach - namely the contingent claim price takes the form of a random walk.**
  2. **There exists a pricing kernel - Back to preferences-based methods of asset pricing.**

Arbitrage and Martingales

- The basic model focuses on pricing the same contingent claim as that of the B-S, except that the risk-free borrowing is not asserted as a possible factor in forming the hedge portfolio.
- Hence the European call option is written on a stock whose pay-off follows $dS = \mu S dt + \sigma S dz$.
- Assuming the current option price takes the form $c(S, t)$, and applying Ito's lemma:

$$dc = \mu_c c dt + \sigma_c c dz$$

$$\mu_c c = c_t + \mu S c_S + \frac{1}{2}\sigma^2 S^2 c_{SS} \qquad \sigma_c c = \sigma S c_S$$

- Following the B-S hedging argument the value of the hedge portfolio is given by $H = -c + c_S S$ (not a zero investment necessarily) with the instantaneous return of $dH = -dc + c_S dS = [c_S \mu S - \mu_c c]dt$.
- The no-arbitrage condition implies:

$$dH = [c_S \mu S - \mu_c c]dt = rHdt = r[-c + c_S S]dt$$

$$\Rightarrow \quad c_S \mu S - \mu_c c = r[-c + c_S S]$$

- Plug $\mu_c c = c_t + \mu S c_S + \frac{1}{2}\sigma^2 S^2 C_{SS}$ in the above condition to get the equilibrium PDE:

$$\frac{1}{2}\sigma^2 S^2 c_{SS} + rSc_S - rc - c_t = 0$$

- Can view this in an alternative way, instead of going through solving the PDE.
- From $\sigma_c c = \sigma S c_S$, can get $C_S = \frac{\sigma_c c}{\sigma S}$.
- Plugging this is the no-arbitrage condition and rearranging, we get:

$$\frac{\mu - r}{\sigma} = \frac{\mu_c - r}{\sigma_c} \equiv \theta(t)$$

which is a new no-arbitrage condition that requires a unique market price of risk $\theta(t)$.

Arbitrage and Martingales

- Can rewrite the stochastic process for the contingent claim as:

$$dc = dc = \mu_c c dt + \sigma_c c dz = [rc + \theta \sigma_c c] dt + \sigma_c c dz$$

- Since $\theta(t)$ is not observable, need to take an approach different than the PDE approach.
- This approach consists a probability measure transformation. Define $d\hat{z}_t = dz_t + \theta(t)dt$ and substitute dz_t in dc to get:

$$dc = rcdt + \sigma_c c d\hat{z}$$

  Risk premium is removed from expected return!
- The probability distribution of future values of c that are generated by $d\hat{z}$ is called the Q probability measure - *The risk-neutral probability measure*.
- It is in contrast to the probability distribution resulted from dz - *The physical probability measure*.

Arbitrage and Martingales

Money market deflator

- Let $B(t)$ be the value of investment in an instantaneous maturity risk-free asset with:

$$\frac{dB}{B} = r(t)dt \;\Rightarrow\; B(T) = B(t)e^{\int_t^T r(u)du}, \;\; \forall\, t \leq T$$

- Define $C(t) \equiv \frac{c(t)}{B(t)}$ is the deflated price process of the contingent claim and apply Ito's lemma to get:

$$dC = \frac{1}{B}dc - \frac{c}{B^2}dB = \frac{rc}{B}dt + \frac{\sigma_c c}{B}d\hat{z} - r\frac{c}{B}dt = \sigma_c C d\hat{z}$$

- As shown, the deflated price process of the contingent claim generated under the Q probability measure is a driftless process. Therefor the expected value of this price for a future date under the Q probability measure equals its current value. The process is a Martingale.

$$C(t) = \hat{E}_t[C(T)] \qquad \forall T \geq t$$

Solution

- Can rewrite the martingale as:

$$\frac{c(t)}{B(t)} = \hat{E}_t[c(T)\frac{1}{B(T)}] = \hat{E}_t[\frac{B(t)}{B(t)e^{\int_t^T r(u)du}}c(T)] = \hat{E}_t[c(T)e^{-\int_t^T r(u)du}]$$

- One can interpret this result as an alternative solution to the B-S equilibrium PDE.
- This says one can value a contingent claim without making any assumptions about the market price of risk if the price is discounted by the risk-free rate factor.

Arbitrage and Pricing Kernels

- Recall that in the two-period/multi-period discrete-time models of consumption-portfolio choice, a risky asset would be priced according to:

$$c(t) = E_t[m_{t,T}c(T)] = E_t[\frac{M_T}{M_t}c(T)] \ , \qquad M_t = U_c(C_t, t)$$

- Does this result hold in continuous time?
- The answer is "Yes" provided the market is dynamically complete.
- To show this, one needs to prove there exists a pricing kernel which satisfies the martingale and no-arbitrage conditions imposed by Black-Scholes model simultaneously.

Arbitrage and Pricing Kernels

- Rewrite the pricing formula as:

$$c(t)M_t = E_t[c(T)M_T]$$

Looks like a martingale!

- Since M_t is the marginal utility, can assume that is follows a strictly positive diffusion process given by:

$$dM = \mu_m dt + \sigma_m dz$$

- Lets impose the no-arbitrage condition.
- Define $c^m \equiv cM$ and apply Ito's lemma to get:

$$dc^m = cdM + Mdc + dMdc = [c\mu_m + M\mu_c c + \sigma_c c\sigma_m]dt + [c\sigma_m + M\sigma_c]dz$$

- cM being a martingale requires that its drift equals zero and therefore:

$$\mu_c = -\frac{\mu_m}{M} - \frac{\sigma_c\sigma_m}{M}$$

Solution

- Applying the last result to the risk-free asset, must impose $\sigma_c = 0$ and set $\mu_c = r(t)$.

$$\Rightarrow \; r(t) = -\frac{\mu_m}{M}$$

- Plugging this result back into the general form of μ_c:

$$\mu_c = r(t) - \frac{\sigma_c \sigma_m}{M} \;\Rightarrow\; \frac{\mu_c - r}{\sigma_c} = -\frac{\sigma_m}{M} = \theta(t)$$

- Now, plugging for μ_m and σ_m in pricing kernel's diffusion process:

$$\frac{dM}{M} = -r(t)dt - \theta(t)dz$$

- Defining $m_t = \ln(M_t)$, $\Rightarrow \; dm = -[r + \frac{1}{2}\theta^2]dt - \theta dz$ and hence,

$$c(t) = E_t[c(T)\frac{M_T}{M_t} = E_t[c(T)e^{m_T - m_t}] = E_t[c(T)e^{-\int_t^T [r(u)+\frac{1}{2}\theta^2(u)]du - \int_t^T \theta(u)dz}]$$