Basics of Asset Pricing Theory
Dynamic Hedging - Derivatives Pricing

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Introduction To Options

- European option: guarantees the right to buy/sell an asset (underlying asset) at a specific future date for a pre-agreed exercise price. The exercise date of a European option is fixed.
  - Call: $c_\tau = \max[S_\tau - X, 0]$
  - Put: $p_\tau = \max[X - P_\tau, 0]$
- American option: guarantees the right to buy/sell an asset (underlying asset) at a specific future date or any time prior to that for a pre-agreed exercise price.
  - Call: $c_t = \max[S_t - X, 0] \ , \quad t \leq \tau$
  - Put: $p_t = \max[X - P_t, 0] \ , \quad t \leq \tau$
- Default-free bound: A financial asset that guarantees payment of a fixed dollar amount at a specific future date T. This date is called the *maturity*. Time to maturity $\tau = T - t$ is the main price determinant of bound prices.

The Black-Scholes Model - Black and Scholes (1973)

Assumptions

- There is a contingent claim whose underlying asset pays no dividends in a frictionless market.
- The underlying asset is a stock whose price (S(t)) follows a diffusion:

$$dS = \mu(S,t)Sdt + \sigma Sdz$$

- There is a riskless investment opportunity such that the amount of riskless investment follows:

$$dB = rBdt$$

- The contingent claim is a European call option written on the stock which matures at T. Its current price, $c(S(t),t)$ is assumed to be twice differentiable in calendar time and the underlying asset's price.

$$c(S(T),T) = \max[S(T) - X, 0]$$

Applying the Ito's lemma:

$$dc = [\frac{\partial c}{\partial S}\mu S + \frac{\partial c}{\partial t} + \frac{1}{2}\frac{\partial^2 c}{\partial c^2}\sigma^2 S^2]dt + \frac{\partial c}{\partial S}\sigma Sdz$$

The hedge portfolio

- Can sell one unit of the call option and hedge the liability by investing in the underlying stock and the risk-free asset. The total investment equaling zero is the restriction to impose.
- Selling one unit of the call option and buying $w(t)$ units of the stock, the surplus or deficit is made up at the risk-free rate.

$$B(t) = c(t) - w(t)S(t) \;\Rightarrow\; dB = [c(t) - w(t)S(t)]rdt$$

- The instantaneous return on the hedge portfolio is given as:

$$dH(t) = -dc(t) + w(t)dS(t) + [c(t) - w(t)S(t)]rdt$$

$$dH(t) = -[\frac{\partial c}{\partial S}\mu S + \frac{\partial c}{\partial t} + \frac{1}{2}\frac{\partial^2 c}{\partial c^2}\sigma^2 S^2]dt - \frac{\partial c}{\partial S}\sigma Sdz + w(t)(\mu(S,t)Sdt + \sigma Sdz) \\ +[c(t) - w(t)S(t)]rdt$$

- Can eliminate the risk by setting $w(t) = \partial c/\partial S$.

No-Arbitrage and PDE

- After risk elimination:

$$dH(t) = [-\frac{\partial c}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c}{\partial S^2} + rc(t) - rS(t)\frac{\partial c}{\partial S}]dt$$

- Frictionless market requires no-arbitrage imposed on any pricing routine. Then, the riskless hedge has to deliver risk-free return. $dH = rHdt$.
- Recall that this hedge took zero investment as the initial value, implying $dH = 0$. Hence,

$$\frac{\partial c}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c}{\partial S^2} - rc(t) + rS(t)\frac{\partial c}{\partial S} = 0$$

- The above expression is a equilibrium partial differential equation connecting the current option price to the underlying asset's price and calendar time.
- Recall the price at maturity, $c(S(T), T) = \max[S(T) - X, 0]$ which serves as a terminal condition for the PDE.

Solution

- Using a separation of variables method (which is not our business here!)

$$c(S(t), t) = S(t)N(d_1) - Xe^{-r(T-t)}N(d_2)$$

with $N(.)$ being a standard normal distribution function and,

$$d_1 = \frac{\ln(S(t)/X) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

- Recall that $w(t) = \partial c/\partial S$. From the above solution $\partial c/\partial S = N(d_1)$ which is a probability and therefore, $0 < w(t) < 1$.
- Finally we can verify the twice differentiability assumption of $c(S(t), t)$ by calculating the first and second derivative from the solution - they exist!

The Vasicek Model - Vasicek (1977)

Assumptions

- There is a "zero-coupon" bond that makes a single payment of \$1 at its maturity date $T = t + \tau$ (τ is time to maturity). It price is hypothesized to be $P(t, \tau)$.
- The instantaneous rate of return on the bond is $\lim_{\tau \to 0} \frac{dP(t,\tau)}{P(t,\tau)} \equiv r(t)dt$
- $r(t)$ follows an Ornstein-Uhlenbeck process (equivalent to a normally distributed AR(1) in discrete time):

$$dr(t) = \alpha[\bar{r} - r(t)]dt + \sigma_r dz_r$$

- Bond prices of all maturity depend on $r(t) \Rightarrow \; P(t, \tau) = P(r(t), \tau)$
- Applying Ito's lemma:

$$dP(r, \tau) = \frac{\partial P}{\partial r} dr + \frac{\partial P}{\partial t} dt + \frac{1}{2} \frac{\partial^2 P}{\partial r^2} (dr)^2$$

$$= [P_r \alpha(\bar{r} - r) + P_t + \frac{1}{2} P_{rr} \sigma_r^2] dt + P_r \sigma_r dz_r$$

The Vasicek Model

Dynamic hedging

- Lets rewrite the price evolution process as:

$$dP(r,\tau) = \mu_p(r,\tau)P(r,\tau)dt - \sigma_p(\tau)P(r,\tau)dz_r$$

- The hedge portfolio consists two different maturity bonds:

$$H(t) = P(r,\tau_1) - \frac{\sigma_p(\tau_1)P(r,\tau_1)}{\sigma_p(\tau_2)P(r,\tau_2)}P(r,\tau_2) = P(r,\tau_1)[1 - \frac{\sigma_p(\tau_1)}{\sigma_p(\tau_2)}]$$

- Instantaneous return of the hedge, by construction, takes the form:

$$dH(t) = dP(r,\tau_1) - \frac{\sigma_p(\tau_1)P(r,\tau_1)}{\sigma_p(\tau_2)P(r,\tau_2)}dP(r,\tau_2)$$

$$= [\mu_p(r,\tau_1) - \frac{\sigma_p(\tau_1)}{\sigma_p(\tau_2)}\mu_p(r,\tau_2)]P(r,\tau_1)dt$$

  which is riskless!

- notice that since default-free bonds are riskless, $r(t)$ has to be the risk-free interest rate.

Dynamic hedging - Market equilibrium

- In the absence of arbitrage, the hedge's instantaneous *rate* of return must equal the risk-free rate:

$$dH(t) = r(t)H(t)dt = r(t)[1 - \frac{\sigma_p(\tau_1)}{\sigma_p(\tau_2)}]P(r,\tau_1)dt$$

$$\Rightarrow \frac{\mu_p(r,\tau_1) - r(t)}{\sigma_p(\tau_1)} = \frac{\mu_p(r,\tau_2) - r(t)}{\sigma_p(\tau_2)}$$

- Sharpe ratios of bonds with different maturities are equal under the no-arbitrage restriction.
- This implies a uniform *market price of interest rate risk* which is independent of time to maturity.

$$\frac{\mu_p(r,\tau) - r(t)}{\sigma_p(\tau)} = q \quad \Rightarrow \quad \mu_p(r,\tau) = r(t) + q\sigma_p(\tau)$$

- Expected return equals the risk-free rate plus a premium proportional to the standard deviation.

The Vasicek Model

Solution

- Substituting back from the Ito's lemma application for $\mu_p(r,\tau)$ and $\sigma_p(\tau)$:

$$P_r\alpha(\bar{r}-r)+P_t+\frac{1}{2}P_{rr}\sigma_r^2=rP-q\sigma_rP_r$$

- Since $\tau=T-t$, $P_t=-P_\tau$, the equilibrium partial differential equation is:

$$\Rightarrow\ \frac{\sigma_r^2}{2}P_{rr}+[\alpha(\bar{r}-r)+q\sigma_r]P_r-rP-P_\tau=0$$

- Can solve the PDE given the boundary condition $P(r,0)=1$ and find the price of the zero-coupon bond as:

$$P(r(t),\tau)=A(\tau)e^{-B(\tau)r(t)}$$

The Merton Model - Merton (1973b)

Assumptions

- This is combination of Black-Scholes and Vasicek models which helps getting closer to reality without making the model too complex to solve.
- All setting are the same as Black-Scholes model.
- The additional aspect is that the risk-free rate evolves randomly over time. This means that, in determining an option's current payoff the present value of the exercise price which changes randomly as well, would be a main factor.
- Let us try pricing a European call option with exercise price of X at its maturity T. Due to randomness of the risk-free rate, one can think of a default-free bond which pays \$$X$ at date T to represent the current value of the option's exercise price.
- Then, for time any maturity $\tau = T - t$ can write the option price as $c(S(t), P(t, \tau), t)$.

The Merton Model

Assumptions

- Like the Vasicek model, the bond's price is given as $(dz_p = -dz - r)$:

$$dP(t,\tau) = \mu_p(t,\tau)P(t,\tau)dt + \sigma_p(\tau)P(t,\tau)dz_p$$

- Bonds and stock prices are correlated: $dz_p dz \equiv \rho dt$
- Having assumed that the option price depend on both stock and bond prices, applying Ito's lemma yields:

$$dc = [\frac{\partial c}{\partial S}\mu S + \frac{\partial c}{\partial P}\mu_p P + \frac{\partial c}{\partial t} + \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\sigma^2 S^2 + \frac{\partial^2 c}{\partial P^2}\sigma_p^2 P^2 + \frac{\partial^2 c}{\partial P \partial S}\rho\sigma\sigma_p SP]dt$$

$$+\frac{\partial c}{\partial S}\sigma S dz + \frac{\partial c}{\partial P}\sigma_p P dz_p$$

$$\equiv \mu_c c dt + \frac{\partial c}{\partial S}\sigma S dz + + \frac{\partial c}{\partial P}\sigma_p P dz_p$$

The Merton Model

Dynamic Hedging

- Can compose a zero investment hedge portfolio like before:
$$c(t) - w_s(t)S(t) - w_p(t)P(t,\tau) = 0$$
- The hedge portfolio's return:

$$\begin{aligned} dH(t) &= -dc(t) + w_s(t)dS(t) + w_p(t)dP(t,\tau) \\ &= [w_s(t)(\mu - \mu_c)S + w_p(t)(\mu_p - \mu_c)P]dt \\ &+ [w_s(t) - \frac{\partial c}{\partial S}]\sigma S dz + [w_p(t) - \frac{\partial c}{\partial P}]\sigma_p P dz_p \end{aligned}$$

- To eliminate risk, must set $w_s(t) = \frac{\partial c}{\partial S}$ and $w_p(t) = \frac{\partial c}{\partial P}$, but according to the hedge portfolio's formation, this means: $c = S\frac{\partial c}{\partial S} + P\frac{\partial c}{\partial P}$.

  Option price is homogeneous of degree 1 in stock and bond prices!

- Lets assume this holds for now. Can verify its accuracy later.

Dynamic Hedging - Market equilibrium

- Like the B-S model, absence of arbitrage implies that instantaneous return on a zero investment portfolio (hedge) must equal zero.

$$w_s(t)(\mu - \mu_c)S + w_p(t)(\mu_p - \mu_c)P = 0$$

$$\Rightarrow \frac{\partial c}{\partial S}(\mu - \mu_c)S + \frac{\partial c}{\partial P}(\mu_p - \mu_c)P = 0 \Rightarrow \frac{\partial c}{\partial S}\mu S + \frac{\partial c}{\partial P}\mu_p P - \mu_c c = 0$$

- Matching the drift of dc with $\mu_c c$ from the above result:

$$-\frac{\partial c}{\partial t} - \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\sigma^2 S^2 - \frac{1}{2}\frac{\partial^2 c}{\partial P^2}\sigma_p^2 P^2 - \frac{\partial^2 c}{\partial S \partial P}\rho\sigma\sigma_p SP = 0$$

$$\Rightarrow \frac{1}{2}[\frac{\partial^2 c}{\partial S^2}\sigma^2 S^2 + \frac{\partial^2 c}{\partial P^2}\sigma_p^2 P^2 + 2\frac{\partial^2 c}{\partial S \partial P}\rho\sigma\sigma_p SP] - \frac{\partial c}{\partial \tau} = 0$$

Equilibrium partial differential equation

Solution

- An interesting observation: The equilibrium PDE doesn't depend on the expected rate of return on the stock, μ, or the expected rate of return on the bond, μ_p.
- There are two boundary conditions needed to solve the PDE, since the option price is subject to two sources of uncertainty over time.

$$c(S(T), P(T,0), T) = \max(S(T) - X, 0)$$

$$P(T,0) = 1$$

- Merton shows the solution is:

$$c(S(t), P(t,\tau), \tau) = S(t)N(h_1) - P(t,\tau)XN(h_2)$$

Homogeneity of degree 1 can be verified!

Derivatives pricing - Summary

- With the underlying asset's price following a diffusion and possibility of continuous trading, dynamic hedging can lead to creation of a totally riskless portfolio against the contingent claim if the -no-arbitrage constraint is imposed.
- Subject to a boundary condition which hold at maturity of the contingent claim and/or the underlying asset(s), one can solve the equilibrium PDE resulted from the no-arbitrage condition.
- The interesting result is that the resulting contingent claim price is independent of underlying assets expected return but rather depends on the volatility. In practice, it is much easier to estimate asset volatilities rather than expected returns.
- No need to assert any consumer problem!