Basics of Asset Pricing Theory

Discrete time consumption-portfolio choice and market equilibrium

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Multi-period consumption-portfolio choice

- The objective function: $E_0[\sum_{t=0}^{T-1} U(C_t, t) + B(W_T, T)]$
- The intertemporal budget constraint:
  $W_{t+1} = (W_t + y_t - C_t)(\sum_{i=1}^{n} \omega_{it} R_{it} + (1 - \sum_{i=1}^{n} \omega_{it}) R_{ft})$

$$\Rightarrow \; W_{t+1} = (W_t + y_t - C_t)(R_{ft} + \sum_{i=1}^{n} \omega_{it}(R_{it} - R_{ft})) \equiv S_t R_t$$

- The consumer maximizes life-time utility by making consumption-saving and investment decisions.
- Investment is made thorough portfolio choices which contain one risk-free and n risky assets.

Multi-period consumption-portfolio choice

Solving the problem:

- Derived utility: $J(W_t, I_t, t) \equiv \max_{C_s, \{\omega_{is}\}} E_t[\sum_{s=t}^{T-1} U(C_s, s) + B(W_T, T)]$ implies that the consumer makes optimal choices at any point in time according to the available set of information.
- So, can solve the problem backwards!
- $J(W_T, T) = E_T[B(W_T, T)] = B(W_T, T)$
- $J(W_T, T-1) = \max_{C_{T-1}, \omega_{i,T-1}} U(C_{T-1}, T-1) + E_{T-1}[B(S_{T-1}R_{T-1}, T)]$
- FOC's:

$$C_{T-1}: \quad U_C(C_{T-1}, T-1) - E_{T-1}[B_W(W_T, T)R_{T-1}] = 0$$

$$\omega_{i,T-1}: \quad E_{T-1}[B_W(W_T, T)(R_{i,T-1} - R_{f,T-1})] = 0, \quad i = 1, ..., n$$

- Plug R_{T-1} in the FOC w.r.t C_{T-1} to get:

Multi-period consumption-portfolio choice

Solving the problem:

$$U_C(C_{T-1}, T-1) = E_{T-1}[B_W(W_T, T)(R_{f,T_1} + \sum_{i=1}^{n} \omega_{i,T-1}(R_{i,T-1} - R_{f,T-1}))]$$

By the FOC's w.r.t $\omega_{i,T-1} \Rightarrow \; U_C(C_{T-1}, T-1) = R_{f,T-1}E_{T-1}[B_W(W_T, T)]$

- The last equation along with the FOC's w.r.t $\omega_{i,T-1}$ wrap up the conditions determining optimal C^*_{T-1} and $\omega^*_{i,T-1}$.
- Applying the envelop theorem: *plugging the optimal values of control variables in the derived utility (value) function* $J(W_{T-1}, T-1)$ *and differentiate w.r.t the state variable* W_T:

$$J_W(W_{T-1}, T-1) = U_C(C^*_{T-1}, T-1)$$

- Recalling $J(W_T, T) = B(W_T, T)$, can rewrite the optimality conditions as:

$$U_C(C^*_{T-1}, T-1) = R_{f,T-1}E_{T-1}[J_W(W_T, T)]$$

$$E_{T-1}[J(W_T, T)(R_{i,T-1} - R_{f,T-1})] = 0, \quad i = 1, ..., n$$

Multi-period consumption-portfolio choice

Solving the problem:

- Optimality principle: *An optimal set of decisions has the property that given an initial decision, the remaining decisions must be optimal with respect to the outcome that results from the initial decision.*
- Then the problem in period $T-2$, $J(W_{T-2}, T-2) = \max\{U(C_{T-2}, T-2) + E_{T-1}[U(C_{T-1}, T-1) + B(W_T, T)]\}$ can be rewritten as:

$$J(W_{T-2}, T-2) = \max_{C_{T-2},\{\omega_{i,T-2}\}} U(C_{T-2}, T-2) + E_{T-2}[\max_{C_{T-1},\{\omega_{i,T-1}\}} E_{T-1}[U(C_{T-1} T-1) + B(W_T, T)]]$$

- Using the previous results:

$$J(W_{T-2}, T-2) = \max_{C_{T-2},\{\omega_{i,T-2}\}} U(C_{T-2}, T-2) + E_{T-2}[J(W_{T-1}, T-1)]$$

Solving the problem:

- FOC's for period $T-2$:

$$C_{T-2}: \quad U_C(C^*_{T-2}, T-2) = J_W(W_{T-2}, T-2)$$

$$\omega_{i,T-2}: \quad E_{T-2}[R_{i,T-2} J_W(W_{T-1}, T-1)] = R_{f,T-2} E_{T-2}[J_W(W_{T-1}, T-1)]$$

- Then, can proceed to the general form - The Bellman equation:

$$J(W_t, t) = \max_{C_t, \omega_{i,t}} U(C_t, t) + E_t[J(W_{t+1}, t+1)]$$

With FOC's:

$$C_t: \quad U_C(C^*_t, t) = E_t[J_W(W_{t+1}, t+1) R_t] = R_{f,t} E_t[J_W(W_{t+1}, t+1)] = J_W(W_t, t)$$

$$\omega_{i,t}: \quad E_t[R_{i,t} J_W(W_{t+1}, t+1)] = R_{f,t} E_t[J_W(W_{t+1}, t+1)], \quad i = 1, ..., n$$

- Depending on the functional form of utility, can solve the system analytically or numerically and find optimal consumption and portfolio as $C^*_t = g[W_t, y_t, I_t, t]$ and $\omega^*_{it} = h[W_t, y_t, I_t, t]$.

Multi-period consumption-portfolio choice

Asset pricing implications:

- By the FOC w.r.t C_t:

$$U_C(C_t^*, t) = R_{f,t} E_t[U_C(C_{t+1}^*, t+1)] \;\Rightarrow\; 1 = R_{f,t} E_t[m_{t,t+1}]$$

- Using the above result, and the FOC w.r.t ω_{it}:

$$U_C(C_t^*, t) = E_t[R_{it} U_C(C_{t+1}^*, t+1)] \;\Rightarrow\; 1 = E[m_{t,t+1} R_{it}]$$

- By recursive substitution of the above equation, one can price trading strategies that can switch assets every period.

$$\begin{aligned} U_C(C_t^*, t) &= E_t[R_{it} E_{t+1}[R_{j,t+1} U_C(C_{t+2}^*, t+2)]] \\ &= E_t[R_{it} R_{j,t+1} U_C(C_{t+2}^*, t+2)] \\ &\Rightarrow\; 1 = E_t[R_{it} R_{j,t+1} m_{t,t+2}] \end{aligned}$$

- The general asset pricing relationship:

$$1 = E_t[R_{t,t+k} m_{t,t+k}]$$

The Lucas Model of Asset Pricing - Lucas (1978)

- Derives the equilibrium prices of risky assets for an *endowment* economy.
- Endowment economy:
  1. The aggregate real output in the economy is generated randomly.
  2. There is no investment/inventory possibilities; The output in non-storable.
  3. Assets are ownership claims on the random output - the output can be viewed as cash dividends.
  4. The aggregate consumption process is fixed. $\Rightarrow$ The economy's stochastic discount factor becomes exogenous.
- The model is built on the multi-period consumption-portfolio choice.
- Impose more structure on asset returns:

$$R_{it} = \frac{d_{i,t+1} + P_{i,t+1}}{P_{it}}$$

- Replacing this stochastic return in $1 = E[m_{t,t+1} R_{it}]$:

$$P_{it} = E_t\left[\frac{U_C(C^*_{t+1}, t+1)}{U_C(C^*_t, t)}(d_{i,t+1} + P_{i,t+1})\right]$$

The Lucas Model of Asset Pricing

- Substituting future period's prices in P_{it} recursively and rearranging terms,

$$P_{it} = E_t[\sum_{j=1}^{T} \frac{U_C(C^*_{t+j}, t+j)}{U_C(C^*_t, t)} d_{i,t+j} + \frac{U_C(C^*_{t+T}, t+T)}{U_C(C^*_t, t)} P_{i,t+T}]$$

- Assuming the time-separable utility has the form $U(C_t, t) = \delta^t u(C_t)$ with $0 < \delta < 1$, the above expression becomes:

$$P_{it} = E_t[\sum_{j=1}^{T} \delta^j \frac{u_C(C^*_{t+j})}{u_C(C^*_t)} d_{i,t+j} + \delta^T \frac{u_C(C^*_{t+T})}{u_C(C^*_t)} P_{i,t+T}]$$

- With $T \to \infty$ the prices converges to:

$$P_{it} = E_t[\sum_{j=1}^{\infty} m_{t,t+j} d_{i,t+j}]$$

- What does determine the asset prices in an endowment economy?

The Lucas Model of Asset Pricing

Examples:

- Risk-neutral agent:

$$P_{it} = E_t[\sum_{j=1}^{\infty} \delta^j d_{i,t+j}]$$

- Log utility - $u(C_t) = ln(C_t)$: Recall that total optimal consumption of an individual each period is set to be equal to aggregated claims to output - $C_t^* = \sum_{i=1}^n d_{it} = d_t$.

$$\Rightarrow \quad P_{it} = E_t[\sum_{j=1}^{\infty} \delta^j \frac{C_t^*}{C_{t+j}^*} d_{i,t+j}] = E_t[\sum_{j=1}^{\infty} \delta^j \frac{d_t}{d_{t+j}} d_{i,t+j}]$$

The market portfolio price is simply computed by summing over assets (i).

$$P_t = E_t[\sum_{j=1}^{\infty} \delta^j \frac{d_t}{d_{t+j}} d_{t+j}] = d_t \frac{\delta}{1-\delta}$$

Future distribution of dividends doesn't matter!

The Lucas Model of Asset Pricing

Examples:

- Power utility (CRRA) - $u(C_t) = C_t^{\gamma}/\gamma$: The market portfolio price takes the form

$$P_t = E_t[\sum_{j=1}^{\infty} \delta^j (\frac{d_{t+j}}{d_t})^{\gamma-1} d_{t+j}] = d_t^{1-\gamma} E_t[\sum_{j=1}^{\infty} d_{t+j}^{\gamma}]$$

Assuming $E_t[d_{t+1}^{\gamma}] = d_{t+1}^{\gamma}$ for simplicity:

$$\Rightarrow \frac{\partial P_t}{\partial d_{t+j}} = \gamma \delta^j (\frac{d_{t+j}}{d_t})^{\gamma-1}$$

Sign of γ determines direction of price movements in response to expected future investment opportunities changes.

d_{it} being non-storable is the key element.

Equity Premium Puzzle

- Recall that $R_r(W) = -(U''(W)/U'(W))W$. For the case of power utility $R_r(W) = 1 - \gamma$.
- Considering an asset in the Lucas model which pays a deterministic dividend of \$1 per period (the risk-free asset), its price takes the form

$$P_{ft} = \frac{1}{R_{ft}} = \delta E_t[(\frac{d_{t+1}}{d_t})^{\gamma-1}]$$

- Mehra and Prescott (1985) try fitting the US data of risk premium based on the above equation and the market portfolio price computed earlier.
- They set d_{it} equal to consumption (assuming the Lucas model works!).
- The interesting result they find is that to make the risk premiums matching the consumption growth requires highly negative magnitudes of γ - extremely risk averse consumers.

This is called the *equity premium puzzle* since then.