Basics of Asset Pricing Theory
Introduction

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The problem: How are financial assets priced?

- The Neoclassical economic theory:

Demand side

- Postulate a utility function of a consumption bundle along with initial endowments for individuals.
- Make some assumptions about consumer's market share. $\Rightarrow$ Price settings.
- Derive the individual and aggregate demand functions (correspondences) given utility maximizing individuals.

Supply side

- Postulate a production function and a cost structure.
- Derive individual and aggregate supply functions (correspondences) given profit maximizing firms.

Equilibrium

- Impose market-clearing conditions.

$\Rightarrow$ Prices are determined in the general equilibrium.

The problem: How are financial assets priced?

- Does this routine work for financial assets? **No!**
- Why?
  1. Consuming assets doesn't generate utility: Who wants to eat Iran-Khodro stocks?
  2. Assets are not (in general) output of some production function: Technology competition is pointless.
  3. Assets are (in general) commitments rather than physical entities: Any one can create an economic commitment: Short sales...
- Instead, assets can be considered as Income/wealth resources. Then, for an individual consumer, as assets' gains rise, consumption possibilities expand.
- Then, we can think of maximizing utility of wealth which indirectly results in consumption utility.

The problem: How are financial assets priced?

- The Asset Pricing theory aims at describing how financial assets are priced using a utility-based approach.
- The analysis follows the assumption that consumers maximize utility of wealth which itself is altered by asset returns.
- On the “supply” side (if we can all it supply), models either rely on stochastic evolution of endowments (dividends), or consider a production economy in which dividends of assets follow stochastic technological changes.
- over the course of this workshop we mostly focus an dynamic models since they are more recent and are actively utilized in research.

The first step is to establish an appropriate utility function.

Expected pay-off vs Expected utility

- Consider an asset offering a random pay-off at a given future date with outcomes $(x_1, ..., x_n)$ and corresponding probability distribution of $(p_1, ..., p_n)$ for states $i = 1, ..., n$ with $\sum_{i=1}^{n} p_i = 1$.
- The expected pay-off is given by $E[\tilde{x}] = \sum_{i=1}^{n} p_i x_i$.
- Is this a good measure of attractiveness for this asset?
- The **St. Petersburg Paradox**: Define an asset with $p_i = (\frac{1}{2})^i$ and $x_i = 2^{(i-1)}$. How much are you willing to pay for this asset?
- The expected value of the above asset is infinite, but in practice people tend to pay a bounded amount.

Bernoulli (1738): Utility of wealth increases in a diminishing rate.

Expected pay-off vs Expected utility

- Therefor, Introducing a concave function of pay-off rather than pay-off itself makes more sense: Expected utility, $E[U(\tilde{x})] = \sum_{i=1}^{n} p_i U_i$, instead of expected pay-off.
- Can we call this a expected utility **function**? Not yet!
- Lets assume all states of the world are reflected in the $x = (x_1, ..., x_n)$ vector. Then, can define a lottery $P = (p_1, ..., p_n)$ as a set of probabilities corresponding to elements of x with $\sum_{i=1}^{n} p_i = 1$.
- If the preferences over all lotteries satisfy *Completeness*, *Transitivity*, *Completeness*, and *Independence*, there exist an expected utility function $V(p_1, ..., p_n)$ representing these preferences (See Chapter 1 for a simple proof).
- This is known as von Neumann - Morgenstern expected utility, with the continuous form of $V(F) = E[U(\tilde{x})] = \int U(x)dF(x)$

Risk-aversion and Pure lotteries

- Define a lottery with $\tilde{\epsilon} = \{\epsilon_1, \epsilon_2\}$, $p = \{p, 1-p\}$ and $E[\tilde{\epsilon}] = 0$.
- Will a risk-averse individual with initial wealth of W take the lottery?
- $U(W) \quad ? \quad E[U(W + \tilde{\epsilon})]$
- $U(W + p\epsilon_1 + (1-p)\epsilon_2) \quad ? \quad pU(W + \epsilon_1) + (1-p)U(W + \epsilon_2)$
- By definition of strict concavity:

$$U(W + p\epsilon_1 + (1-p)\epsilon_2) > pU(W + \epsilon_1) + (1-p)U(W + \epsilon_2)$$

A risk-averse individual always refuses a fair lottery.

- The above argument rationalizes existence of risk premium for risky assets.

$$U(W - \pi) = E[U(W + \tilde{\epsilon})]$$

Measuring risk aversion

- One can solve for π using Taylor expansions of the two sides of the previous expression. So that with $\sigma^2 \equiv E[\tilde{\epsilon}^2]$:

$$\pi = -\frac{1}{2}\sigma^2 \frac{U''(W)}{U'(W)}$$

- Measure of absolute risk aversion: $R(W) \equiv \frac{U''(W)}{U'(W)}$
- Measure of relative risk aversion: $R_r(W) = WR(W)$

A two-period model of consumption-portfolio choice

- The consumer lives two periods and can invest in i risky assets in date 0 to gain random returns in date 1. The random pay-off of an asset can be represented by $X_i = \tilde{P}_{1i} + \tilde{D}_{1i}$.
- The consumer maximizes the life-time utility by making a consumption-saving decision on date zero resources, and an investment decision which picks the optimal portfolio weights distribution over the resources allocated to saving: $\max_{C_0, \{\omega_i\}} U(C_0) + \delta E[U(C_1)]$
- The intertemporal budget constraint: $C_1 = y_1 + (W_0 + y_0 - C_0) \sum_{i=1}^{n} \omega_i R_i$
- Portfolio weights must sum up to unity: $\sum_{i=1}^{n} \omega_i = 1$ (Possibility of short sales).

A two-period model of consumption-portfolio choice

- FOC's:

$$C_0: \qquad U'(C_0) - \delta E[U'(C_1)\sum_{i=1}^{n}\omega_i R_i] = 0$$

$$\omega_i: \qquad \delta E[U'(C_1)R_i] - \lambda = 0, \quad i = 1, ..., n$$

- **Result 1**: For any two different assets i and j, $E[U'(C_1)R_i] = E[U'(C_1)R_j]$. The choice between assets is subject to trade-off as long as they generate different returns.
- **Result 2**: $\delta E[U'(C_1)R_i] = U'(C_0), \quad i = 1, ..., n$ from FOC's. Also $R_i = \frac{X_i}{P_i}$. Then asset prices are determined as:

$$P_i = E[\frac{\delta U'(C_1)}{U'(C_0)X_i}] = E[m_{01}X_i]$$

A two-period model of consumption-portfolio choice

Asset pricing implications:

- **Risk-free rate**: $U'(C_0) = R_f \delta E[U'(C_1)] \Rightarrow 1/R_f = E[m_{01}]$
- **Risk premium**: From Result 2

$$1 = E[m_{01} R_i] = E[m_{01}](E[R_i] + \frac{Cov(m_{01}, R_i)}{E[m_{01}]})$$

$$\Rightarrow R_f = E[R_i] + \frac{Cov(m_{01}, R_i)}{E[m_{01}]} \Rightarrow E[R_i] = R_f - \frac{Cov(U'(C_1), R_i)}{E[U'(C_1)]}$$

- **CAPM**: Suppose there is a portfolio with random return of $\tilde{R}_m$ such that $U'(\tilde{C}_1) = -\kappa \tilde{R}_m, \quad \kappa > 0$. Then $Cov[U'(C_1), R_m] = -\kappa Var[R_m]$ and $Cov[U'(C_1), R_i] = -\kappa Cov[R_m, R_i]$.

$$\Rightarrow \frac{E[R_m] - R_f}{E[R_i] - R_f} = \frac{\kappa Var[R_m]}{\kappa Cov(R_m, R_i)}$$

$$\Rightarrow E[R_i] - R_f = \beta_i (E[R_m] - R_f)$$

Asset pricing implications:

- **Hanson-Jagannathon bound**: Given the derived risk premium for asset i:

$$E[R_i] - R_f = -\frac{Cov(U'(C_1), R_i)}{E[U'(C_1)]} \Rightarrow E[R_i] - R_f = -\rho_{m_{01},R_i}\frac{\sigma_{m_{01}}\sigma_{R_i}}{E[m_{01}]}$$

$$\Rightarrow \frac{E[R_i] - R_f}{\sigma_{R_i}} = -\rho_{m_{01},R_i}\frac{\sigma_{m_{01}}}{E[m_{01}]}$$

Since $-1 < \rho_{m_{01},R_i} < 1$,

$$\Rightarrow \left|\frac{E[R_i] - R_f}{\sigma_{R_i}}\right| \leq \frac{\sigma_{m_{01}}}{E[m_{01}]} = \sigma_{m_{01}} R_f$$